ON THE RATIONAL PICARD GROUP OF THE MODULI SPACE OF HIGHER SPIN CURVES

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Abstract – We refine the notion of higher spin curves defined in [CCC07] in terms of line bundles, by adding the data of coherent nets of roots introduced by [Jar01] in terms of torsion-free sheaves and we describe the boundary part of their moduli space $\overline{S}_{g,n}^{\frac{1}{r},\mathbf{m}}$. We provide then a presentation of the rational Picard group of $\overline{S}_{g,n}^{\frac{1}{r},\mathbf{m}}$.

Key words: Higher spin curves; Moduli of curves; Picard group.

1. Introduction

In [Jar01], the moduli space of higher spin curves with the additional data of *coherent net of roots* is described by using torsion-free sheaves. The aim of this note is to refine the notion of root given in the line bundles setting by [CCC07] with such additional data and investigate the geometry of the corresponding moduli space. This space is less singular than the one described in [CCC07] and it is thus possible to study its Picard group.

A stable spin curves of type $\mathbf{m}$ is an n-pointed curves of genus g with only ordinary nodes as singularities together with the data of a coherent net of rth roots of type $\mathbf{m}$ on it. The set of stable spin curves up to isomorphisms is in one-to-one correspondence with the moduli space $\overline{\mathfrak{S}}_{g,n}^{\frac{1}{r},\mathbf{m}}$ of higher spin curves described by [Jar01], hence we can transport the scheme structure of the latter to the former and consider the two spaces as isomorphic schemes (see Section 2).

In order to describe the rational Picard group of this moduli space, we restrict to one irreducible component whenever $S_{g,n}^{\frac{1}{r},\mathbf{m}}$ is not irreducible. The low dimensional homology and cohomology of the open part $S_{g,n}^{\frac{1}{r},\mathbf{m}}$ has been studied in [RW12], together with the relations between boundary divisors and other classes, such as the classes λ and μ. It turns out that, when the genus g is greater or equal to 9, the first holomology group over $\mathbb{Q}$ is zero, while the second cohomology group has rank 1. This means that, for instance, the Hodge class generates the Picard group $\mathrm{Pic}(S_{g,n}^{\frac{1}{r},\mathbf{m}})$ over $\mathbb{Q}$.

By [Jar01], the classes λ, $\alpha_i^{(a,b)}$ and $\gamma_{j,n}^{(a,b)}$, where the $\alpha_i^{(a,b)}$'s and $\gamma_{j,n}^{(a,b)}$'s denote suitable boundary divisors (to be defined in Section 3), are independent in $\mathrm{Pic}(\overline{S}_{g,n}^{\frac{1}{r},\mathbf{m}})$. By [Jar00] and [Jar01] the space $\overline{S}_{g,n}^{\frac{1}{r},\mathbf{m}}$ is normal and with quotient singularities, so that, as in [BF06], its rational Picard group is isomorphic to the correspondent Chow group over $\mathbb{Q}$. This implies that the whole Picard group $\mathrm{Pic}(\overline{S}_{g,n}^{\frac{1}{r},\mathbf{m}})$ is generated over $\mathbb{Q}$ by the generators of the Chow group of the open part $S_{g,n}^{\frac{1}{r},\mathbf{m}}$ together with the set of boundary classes of $\overline{S}_{g,n}^{\frac{1}{r},\mathbf{m}}$. In the end we obtain a complete description of the generators of the rational Picard group (see Section 4):

Theorem 4.3. *Assume $g \geq 9$. Then $\mathrm{Pic}(\overline{S}_{g,n}^{\frac{1}{r},\mathbf{m}})$ is freely generated over $\mathbb{Q}$ by the classes λ, $\alpha_i^{(a,b)}$ and $\gamma_{j,\eta}^{(a,b)}$, where λ is the Hodge class and $\alpha_i^{(a,b)}$ and $\gamma_{j,\eta}^{(a,b)}$ are the boundary divisors.*

In the next future, we plan to produce a presentation of the integral Picard group of $\overline{S}_{g,n}^{\frac{1}{r},\mathbf{m}}$ by generators and relations.

We work over the complex field $\mathbb{C}$.

2. The moduli space

Let C be an n-pointed curve of genus g with only ordinary nodes as singularities, $\mathbf{m} = (m_1, \ldots, m_n)$ an n-tuple of integers and $r \in \mathbb{N}$ a positive integer dividing $2g-2-\sum m_i$. Following [CCC07], we define a *rth-root* as follows.

Definition 2.1 (r-th root of type $\mathbf{m}$)**.** *Let $(C,(q_1,\ldots,q_n))$ be an n-pointed curve with only ordinary nodes as singularities. We say that $((X,(h_1,\ldots,h_n)),L,\alpha)$ is an r-th root of $((C,(q_1,\ldots,q_n)),\omega_C)$ of type $\mathbf{m}$ if*

- *$\pi : (X,(h_1,\ldots,h_n)) \to (C,(q_1,\ldots,q_n))$ is a blow-up of C at certain nodes $\{p_i\}$,*
- *$L \in \mathrm{Pic}(X)$*
- *$\alpha : L^{\otimes r} \to \pi^*\omega_C(-\sum m_k q_k)$ is an homomorphism satisfying*
  - *$L_{|E_i}$ has degree 1,*
  - *the map α is an isomorphism outside the exceptional components*
  - *for every exceptional component E_i, the order of vanishing u_i and v_i of α at $\{p_i^+, p_i^-\} = \pi^{-1}(p_i)$ add up to r,*

Let Δ be the set of nodes that are blown up in $\pi : X \to C$. Similarly to the situation discussed in [CCC07] for $\mathbf{m} = \mathbf{0}$, if (X,L,α) is an rth-root of (C,ω_C) of type $\mathbf{m} \neq \mathbf{0}$ and if $\widetilde{\pi} : \widetilde{X} \to C$ denotes the partial normalization at Δ, then

$$\widetilde{L}^{\otimes r} = \widetilde{\pi}^*(\omega_C)\left(-\sum_i (u_i p_i^+ + v_i p_i^-) - \sum_k m_k q_k\right) \tag{1}$$

and the homomorphism α is defined to agree with

$$\widetilde{L}^{\otimes r} = \widetilde{\pi}^*(\omega_C)\left(-\sum (u_i p_i^+ v_i p_i^-) - \sum_k m_k q_k\right) \hookrightarrow \widetilde{\pi}^*(\omega_C)\left(-\sum_k m_k q_k\right)$$

on $\widetilde{X}$ and to be zero on all exceptional components.

Now we define the analogue of a *coherent net of roots* (see [Jar01]) for this situation.

Let d be an integer such that $d|r$ and define $\mathbf{m}^{\mathrm{d}}$ as the n-tuple given by

$$m_i^d := \min\{m \in \mathbb{N} \ : \ m \equiv m_i \mod d\}\,.$$

Definition 2.2 (Coherent net of roots)**.** *Let C and X be as above. A* coherent net of roots *of type $\mathbf{m}$ for ω_C is the data $((X,(h_1,\ldots,h_n)),\{L_d,\alpha_{d,d'}\})$ where $\{L_d,\alpha_{d,d'}\}$, is a collection consisting of a line bundle L_d on X for each $d|r$ and a homomorphism*

$$\alpha_{d,d'} : L_d^{\otimes \frac{d}{d'}} \to L_{d'} \qquad \forall d'|d$$

such that

- *$L_1 = \pi^*\omega_C(-\sum_k m_k q_k)$ and $\alpha_{d,d} = \mathrm{id}$ for each $d|r$,*
- *for every $d|r$, the map $\alpha_{d,1}$ makes $(X,L_d,\alpha'_{d,1})$ into a dth-root of (C,ω_C) of type $\mathbf{m}^d + \mathbf{m}$, where $\alpha'_{d,1}$ is given by*

  $$\alpha'_{d,1} : L_d^{\otimes d} \to \pi^*\omega_C\left(-\sum (m_k + m_k^d) q_k\right) = L_1 \otimes \mathcal{O}_X\left(-\sum m_k^d q_k\right)$$

- *for every $d'|d|r$, the line bundle homomorphism $\alpha_{d,d'}$ makes the following diagram commute for every $d''|d'|d|r$:*

$$\begin{tikzcd} \left(L_d^{\otimes \frac{d}{d'}}\right)^{\otimes \frac{d'}{d''}} \arrow[r, "\alpha_{d,d'}^{\otimes \frac{d'}{d''}}"] \arrow[dr, "\alpha_{d,d''}"'] & L_{d'}^{\otimes \frac{d'}{d''}} \arrow[d, "\alpha_{d',d''}"] \\ & L_{d''} \end{tikzcd}.$$

Let us consider for a moment the case $\mathbf{m} = \mathbf{0}$. Roughly speaking, a coherent net of roots corresponds to an rth root L of ω_C plus the dth roots $L^{\otimes r/d}$ for each d dividing r. In other words, a coherent net of roots corresponds to

- a dth root $(X, L_d, \alpha_{d,1})$ of (C, ω_C) for every $d|r$,
- a line bundle homomorphism $\alpha_{d,d'} : L_d^{\otimes d/d'} \to L_{d'}$ for every $d'|d|r$, such that each diagram as in the definition commutes.

The line bundle homomorphism $\alpha_{d,d'}$ does not give a d/d'th root, since a root is defined only for a line bundle in $\mathrm{Pic}(C)$. But $\pi_*(L_{d'})$ (the one that would be the natural choice) is not in general a line bundle on C, since it may have torsion. Consequently, $\pi^*\pi_*(L_{d'})$ is not in general a line bundle on X. This is the reason why, in the definition, we not ask as in [Jar01] that every L_d is a d/d'-th root of $L_{d'}$ but we only requires the commutativity of the diagrams. By the way, notice that there is a misprint in the Definition 2.4 of [Jar01], which should be corrected as follows [Jar] : if $\mathcal{E}_d$ and $\mathcal{E}_{d'}$ correspond to L_d and $L_{d'}$, than $\mathcal{E}_d$ should not be a d/d' root of $\mathcal{E}_{d'}$ of type $\mathbf{m}^{d/d'}$ but rather of type $\frac{d'}{d}(\mathbf{m}^d - \mathbf{m}^{d'})$.

As in [Jar01], we can define isomorphisms of coherent nets of rth roots.

Definition 2.3. *An* isomorphism of coherent nets of rth roots *from* $((X, (h_1, \dots, h_n)), \{L_d, \alpha_{d,d'}\})$ *to* $((X', (h'_1, \dots, h'_n)), \{L'_d, \alpha'_{d,d'}\})$ *is an isomorphism of pointed curves*

$$\tau : (X, (h_1, \dots, h_n)) \to (X', (h'_1, \dots, h'_n))$$

together with a system of isomorphisms $\{\beta_d : \tau^* L'_d \to L_d\}$ *compatible with all the maps* $\tau^*\alpha'_{d,d'}$ *and* $\alpha_{d,d'}$ *and such that* β_1 *is the canonical isomorphism*

$$\beta_1 : \tau^*\pi'^*\omega_C\left(-\sum_k m_k q_k\right) \to \pi^*\omega_C\left(-\sum_k m_k q_k\right).$$

In order to relate this construction to the one of [Jar01], we recall a well-known result about line bundles and torsion-free sheaves of rank one.

Proposition 2.4 ([CCC07], Proposition 4.2.2.)**.** *Let B an integral scheme and $f : \mathcal{C} \to B$ and $f : \mathcal{C} \to B$ a family of nodal curves.*

(1) *Let $\pi : \mathcal{X} \to \mathcal{C}$ be a family of blow-ups of $\mathcal{C}$ and let $\mathcal{L} \in \mathrm{Pic}\mathcal{X}$ be a line bundle having degree 1 on every exceptional component. Then $\pi_*(\mathcal{L})$ is a relatively torsion-free sheaf of rank 1, flat over B.*

(2) *Conversely, suppose that $\mathcal{F}$ is a relatively torsion-free sheaf of rank 1 on $\mathcal{C}$, flat over B. Then there exist a family $\pi : \mathcal{X} \to \mathcal{C}$ of blow-ups of $\mathcal{C}$ an a line bundle $\mathcal{L} \in \mathrm{Pic}\mathcal{X}$ having degree 1 on all exceptional components, such that $\mathcal{F} \cong \pi_*(\mathcal{L})$.*

(3) *Let $\pi : \mathcal{X} \to \mathcal{C}$ and $\pi' : \mathcal{X}' \to \mathcal{C}$ be families of blow-ups of $\mathcal{C}$ and $\mathcal{L} \in \mathrm{Pic}\mathcal{X}$, $\mathcal{L}' \in \mathrm{Pic}\mathcal{X}'$ line bundles having degree 1 on every exceptional component. Then*

$$\pi_*(\mathcal{L}) \cong \pi'_*(\mathcal{L}') \iff \exists \sigma : \mathcal{X} \xrightarrow{\sim} \mathcal{X}'$$
$$\text{isomorphism over } \mathcal{C} \text{ s.t. } \mathcal{L} \cong \sigma^*(\mathcal{L}').$$

Let us consider the coarse moduli space $\overline{\mathfrak{S}}_{g,n}^{\frac{1}{r},\mathbf{m}}$ defined by Jarvis in [Jar01] and the set $\overline{S}_{g,n}^{\frac{1}{r},\mathbf{m}}$ of all isomorphism classes of stable spin curves of type $\mathbf{m}$, *i.e.* n-pointed curves of genus g together with the data of a coherent net of rth roots of type $\mathbf{m}$ on it.

We restrict for simplicity to the case of curves without marked points. As observed in [CCC07], if $(\mathcal{E}, \beta)$ is an rth root of (C, ω_C) in the sense of [Jar01], then $\mathcal{E}$ is always of the form $\pi_*(L)$ for some $L \in \mathrm{Pic}(X)$, where X is the curve obtained from

C by blowing-up the nodes where $\mathcal{E}$ is not locally free, $\pi : X \to C$ is the natural morphism and $(X, L, \overline{\beta})$ is an rth root of (C, ω_C).

Considering a coherent net of rth-roots $\{\mathcal{E}_d, \beta_{d,d'}\}$, the same clearly holds for $\mathcal{E}_r$ and $\beta_{r,1}$, where X is the curve obtained from C by blowing-up the nodes where $\mathcal{E}_r$ is not locally free, and we thus get a line bundle $L_r \in \mathrm{Pic}(X)$ satisfying $\pi_*(L_r) \cong \mathcal{E}_r$. Furthermore, by Proposition 2.4, this curve X is unique up to isomorphisms. If we now consider all coherent nets of sheaf rth-roots $\{\mathcal{F}_d, \gamma_{d,d'}\}$ with $\mathcal{F}_r = \mathcal{E}_r$ and $\gamma_{r,1} = \beta_{r,1}$, they are clearly in one-to-one correspondence with the coherent nets of line bundle rth-roots $(X, \{F_d, f_{d,d'}\})$ with $F_r = L_r$ and $f_{r,1} = \overline{\beta}_{r,1}$, up to isomorphisms (since coherent nets of roots are defined essentially in the same way in the two situations).

We can conclude that there is a one-to-one set-theoretical correspondence between the coarse moduli space $\overline{\mathfrak{S}}^{\frac{1}{r},\mathbf{m}}_{g,n}$ constructed with coherent nets of sheaves roots and the set $\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}$ constructed with coherent nets of line bundles roots. Thanks to this correspondence, we can transport the scheme structure of the space $\overline{\mathfrak{S}}^{\frac{1}{r},\mathbf{m}}_{g,n}$ to the space $\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}$, hence consider the two spaces as isomorphic schemes.

3. The boundary divisors

Since the two schemes $\overline{\mathfrak{S}}^{\frac{1}{r},\mathbf{m}}_{g,n}$ and $\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}$ are isomorphic, the description of the boundary divisor in $\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}$ comes straightforward from the one of $\overline{\mathfrak{S}}^{\frac{1}{r},\mathbf{m}}_{g,n}$ made in [Jar01], which we recall here.

Example 1. Let C be a stable curve with two smooth, irreducible components D and F of genus k and $g-k$ respectively, meeting in one double point p. Let $(X, \{L_r, \alpha_{d,d'}\})$ be a coherent net of rth-roots for (C, ω_C) and let $\{u, v\}$ be the order of vanishing of $\alpha_{r,1} : L_r^{\otimes r} \to \pi^*\omega_C$ at p.

By [CCC07] or [Jar01], outside the exceptional component we have

$$\widetilde{L}_r^{\otimes r} = \widetilde{\pi}^*\omega_C(-up^+ - vp^-) = \omega_{\widetilde{X}}(-(u-1)p^+ - (v-1)p^-),$$

where $\widetilde{\pi} : \widetilde{X} \to C$ is the partial normalization at p, $\widetilde{L}$ is the lifting of L to $\widetilde{X}$ and $\{p^+, p^-\} = \pi^{-1}(p)$. We must have the degree of $\widetilde{\pi}^*\omega_C(-up^+ - vp^-)$ divisible by r and this implies there exists a unique choice possible for u (and $v = r - u$), given by (see for instance [CCC07, p. 29])

$$u \equiv \deg \omega_D = 2k - 2 + 1 \mod r, \qquad v \equiv \deg \omega_F = 2g - 2k - 2 + 1 \mod r.$$

If $u = 2k - 1 \not\equiv 0 \mod r$, then the resulting rth root corresponds to an rth root of $\omega_D(-(u-1)p^+)$ on D and an rth root of $\omega_F(-(v-1)p^-)$ on F. As explained in [Jar01], this completely determines also all other L_d's of the net (up to isomorphism), hence the spin curve is represented by an element in $S^{\frac{1}{r},u-1}_{k,1} \times S^{\frac{1}{r},v-1}_{g-k,1}$.

Example 2. Let now be C an irreducible stable curve with one node. In this case there are r different possible choices for u and v: besides the trivial case $u = v = 0$, the cases $u = 1, \dots, r-1$ and $v = r - u$. In this case, if u and v a relatively prime, we know that the spin structure is induced by

$$\widetilde{L}^{\otimes r} = \widetilde{\pi}^*\omega_C(-up^+ - vp^-) = \omega_{\widetilde{X}}(-(u-1)p^+ - (v-1)p^-)$$

where $\widetilde{L}$ is, by definition, an rth root of $\omega_{\widetilde{X}}(-(u-1)p^+ - (v-1)p^-)$ and it thus corresponds to an element of $S^{\frac{1}{r},(u-1,v-1)}_{g-1,2}$. Since the points p^+ and p^- on $\widetilde{X}$ are not ordered, there is a degree-2 morphism $\overline{S}^{\frac{1}{r},(u-1,v-1)}_{g,2} \to \overline{S}^{\frac{1}{r}}_{g}$. If $(u, v) = l > 1$,

then for each $\widetilde{L}_l$ there are l distinct choices of gluing data and l choices of L_l. This gives $l/2$ distinct morphisms from $\overline{S}_{g,2}^{\frac{1}{r},(u-1,v-1)}$ to $\overline{S}_g^{\frac{1}{r}}$.

Recall that the boundary divisors of $\overline{M}_g$ consist of the divisor δ_0, whose general member corresponds to an irreducible curve with one node, together with the divisors δ_i with $i = 1, \ldots, [g/2]$, where two irreducible components meet in one node.

Let $l_{g,r}(\mathbf{m})$ be defined (see [Jar01]) by

$$l_{g,r}(\mathbf{m}) := \begin{cases} 1 & g = 0, r|2 + \sum m_i \\ \gcd(r, m_1, \ldots, m_n) & g = 1, r|\sum m_i \\ \gcd(r, 2, m_1, \ldots, m_n) & g > 1, r|\sum m_i + 2 - 2g \\ 1 & \text{otherwise} \end{cases}$$

and let $D_{g,r}(\mathbf{m})$ denote the set of positive divisors of $l_{g,r}(\mathbf{m})$. By Theorem 2.7 of [Jar01], this set is in one to one correspondence with the set of irreducible components of $\overline{S}_{g,r}^{\frac{1}{r},\mathbf{m}}$. Note that for $g > 1$, the integer $l_{g,r}(\mathbf{m})$ can take only the values 1 or 2, depending on whether r is odd or even.

We have seen before in Example 1 that for any curve X with exactly one node and two irreducible components of genera i and $g - i$ there is a unique choice of an integer $u(i) \in \{1, \ldots, r-1\}$ that determines the spin structure on the curve. For each i between 1 and $g-1$ and for each $a \in D_{i,r}(u(i)-1)$ and $b \in D_{g-i,r}(v(i)-1)$, let $\alpha_i^{(a,b)}$ denote the irreducible divisor consisting of the locus of spin curves lying over $\delta_i \in \operatorname{Pic}(\overline{M}_g)$ with a spin structure of index a on the genus-i component and of index b on the genus-$(g-i)$ component of the general member of δ_i.

Consider now the divisors lying over δ_0. For any choice of $\{u, v\}$ we know that any rth root of the bundle $\omega_{\widetilde{X}}(-(u-1)p^+ - (v-1)p^-)$ determines the entire spin structure except for the glueing and, conversely, any spin structure comes from an rth root of a bundle like that one for some $u \in \{0, \ldots, r-1\}$ and a choice of glueing (which corresponds to an isomorphism $\eta : L_{l|p^+} \xrightarrow{\sim} L_{l|p^+}$ with $l = \gcd(u, v, r)$). Denote by $\gamma_{j,\eta}^{(a)}$ the irreducible divisor with order $j := \min(u, v)$, index $a \in D_{g-1,r}(u-1, v-1)$ and glueing η. Clearly $\gamma_{j,\eta}^{(a)} = \gamma_{r-j,\eta^{-1}}^{(a)}$.

By [Jar01, Proposition 3.4], the set of all this boundary divisors together the Hodge class λ (for its definition see, for instance, [Jar01, §3.2]) is a set of indipendent elements in the Picard group of $\overline{S}_g^{\frac{1}{r}}$ whenever $g > 1$.

4. The main result

Let $\mathcal{K}$ be a line bundle on the universal curve $\mathcal{C}_{g,n}$ over the stack of stable curves. Let us define $\overline{\mathfrak{S}}_{g,n}^{\frac{1}{r},\mathbf{m}}(\mathcal{K})$ as in [Jar00], to be the stack of genus g, n-pointed, stable curves, together with the data of a coherent net of rth-roots of $\mathcal{K}$ of type $\mathbf{m}$ on the curve.

Proposition 4.1. *The space $\overline{\mathfrak{S}}_{g,n}^{\frac{1}{r},\mathbf{m}}(\mathcal{K})$ is a smooth, proper, Deligne-Mumford stack over $\mathbb{Z}[1/r]$ and its coarse moduli space $\overline{\boldsymbol{\mathfrak{S}}}_{g,n}^{\frac{1}{r},\mathbf{m}}(\mathcal{K})$ has quotient singularities.*

Proof. The first part of the Proposition is Theorem 2.4.4 of [Jar00], which states also that the natural forgetful morphism

$$\overline{\mathfrak{S}}_{g,n}^{\frac{1}{r},\mathbf{m}}(\mathcal{K}) \to \overline{\mathcal{M}}_{g,n}$$

is finite and surjective. Next, according to Theorem 2.7 of [Jar01], the coarse moduli space of $\overline{\mathfrak{S}}_{g,n}^{\frac{1}{r},\mathbf{m}}(\mathcal{K})$ is normal and projective. By Proposition 2.8 of [Vis89], if a scheme of finite type over a field of characteristic zero is the moduli space of some smooth stack, then its normalization has quotient singularities. Since the coarse

moduli space of $\overline{\mathfrak{S}}^{\frac{1}{r},\mathbf{m}}_{g,n}(\mathcal{K})$ is itself normal, we can conclude that it has quotient singularities. □

Let us focus on the case $\mathcal{K} = \omega$, where ω is the relative dualizing sheaf of the universal curve $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, and drop the "$\omega$" in the notation. By our remark at the end of Section 1, we may conclude that also the scheme $\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}$ has only quotient singularities. Hence, the analogue of Lemma 1 of [BF06] holds.

Lemma 4.2. *Let* $\mathrm{Pic}(\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n})$ *be the Picard group of* $\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}$*. Then there is a natural isomorphism*

$$\mathrm{Pic}(\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}) \otimes \mathbb{Q} \cong A_{3g-4+n}(\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}) \otimes \mathbb{Q}.$$

Proof. Since $\overline{\mathfrak{S}}^{\frac{1}{r},\mathbf{m}}_{g,n}$ is normal (see Theorem 2.7 of [Jar01]) and the scheme structure of $\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}$ is by definition the one of $\overline{\mathfrak{S}}^{\frac{1}{r},\mathbf{m}}_{g,n}$, the moduli $\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}$ is also normal and there is an injection

$$\mathrm{Pic}(\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}) \hookrightarrow A_{3g-4+n}(\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}).$$

Moreover, we know that $\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}$ has only quotient singularities, hence every Weil divisor is a $\mathbb{Q}$-Cartier divisor. We thus obtain a surjection

$$\mathrm{Pic}(\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}) \otimes \mathbb{Q} \twoheadrightarrow A_{3g-4+n}(\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}) \otimes \mathbb{Q},$$

hence our claim follows. □

We are now ready to state our main result.

Theorem 4.3. *Assume* $g \geq 9$*. Then* $\mathrm{Pic}(\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n})$ *is freely generated over* $\mathbb{Q}$ *by the classes* λ*,* $\alpha_i^{(a,b)}$ *and* $\gamma_{j,\eta}^{(a,b)}$*, where* a, b, i, j *and* η *are as described in the previous section.*

Proof. In order to gain a description of the generators of $\mathrm{Pic}(\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}) \otimes \mathbb{Q}$ it is enough to describe $A_{3g-4+n}(\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}) \otimes \mathbb{Q}$. Indeed, as in [BF06], thanks to the exact sequence

$$A_{3g-4+n}(\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n} \setminus S^{\frac{1}{r},\mathbf{m}}_{g,n}) \to A_{3g-4+n}(\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}) \to A_{3g-4+n}(S^{\frac{1}{r},\mathbf{m}}_{g,n}) \to 0,$$

the group $\mathrm{Pic}(\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}) \otimes \mathbb{Q}$ is generated by the generators of $A_{3g-4+n}(S^{\frac{1}{r},\mathbf{m}}_{g,n})$ together with the set of boundary classes of $\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}$.

Consider first the open part and restrict to one irreducible component whenever $S^{\frac{1}{r},\mathbf{m}}_{g,n}$ is not irreducible. To do this, denote by $S^{\frac{1}{r},\mathbf{m}}_{g,n}[\epsilon]$ the whole $S^{\frac{1}{r},\mathbf{m}}_{g,n} \otimes_{\mathbb{Z}[1/r]} \mathbb{C}$ if r is odd or the component of Arf invariant ϵ if r is even. Theorem 1.4 of [RW12] shows that for $g \geq 6$ it is

$$H_1(S^{\frac{1}{r},\mathbf{m}}_{g,n}[\epsilon]; \mathbb{Q}) = 0$$

and for $g \geq 9$ the second cohomology group $H^2(S^{\frac{1}{r},\mathbf{m}}_{g,n}[\epsilon]; \mathbb{Q})$ has rank 1 and it is thus generated by only one class. Since the Hodge class λ is a non trivial class in this group, we can conclude that the open part $\mathrm{Pic}(S^{\frac{1}{r},\mathbf{m}}_{g,n}[\epsilon]) \otimes \mathbb{Q}$ is generated, for instance, by λ.

Now we take care of the contributions coming from the boundary part. By [Jar01] the $\{\alpha_i^{(a,b)}\}$ and the $\{\gamma_{j,n}^{(a,b)}\}$ are generators for the boundary divisors of the space $\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n}$. Moreover, by Proposition 3.4 of [Jar01], the classes λ, $\alpha_i^{(a,b)}$ and $\gamma_{j,n}^{(a,b)}$ are independent in $\mathrm{Pic}(\overline{\mathfrak{S}}^{\frac{1}{r},\mathbf{m}}_{g,n})$ for $g > 1$, hence they are independent also in $\mathrm{Pic}(\overline{S}^{\frac{1}{r},\mathbf{m}}_{g,n})$ and the claim follows.

□

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